Equation (8) is the solution to the MHD Couette flow formation from system starting at rest. The solution to this problem was solved by Tao.[2] However, Eq. (8) is much more compact and lends itself much more easily to physical interpretation and calculation than does the solution presented by Tao. Equation (8) shows that for large ω the fluid does not have time enough to respond fully to the changing magnetic field and responds only to the rms value.

A sample calculation was performed for the following conditions:

$$\begin{aligned} \sigma &= 25 \text{ mho/m} \\ h &= 0.1 \text{ m} \\ \mu &= 1 \text{ cp} \\ B &= 6350 \text{ gauss} \\ N_H &= 10 \\ \omega &= 0.1 \text{ cps} \end{aligned}$$

After 500 sec, the system will reach an almost steady state that, for all practicality, is nonoscillating in the velocity field and is uniformly oscillating in the current field $\mathbf{J} = \sigma \mathbf{u} \times \mathbf{B}$. The actual current distortion is less than 5% of the peak value, thus indicating that a value of $\omega = 0.1$ cps is still "large." It is interesting to note that the distortion of the a.c. field is minimized for both $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. When the fluid velocity field partially follows (lags behind) the magnetic field, distortion in the a.c. field arises.

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Inner-Outer Expansion Method for Couple-Stress Effects

S. H. Lam* and D. C. Leigh†
Princeton University, Princeton, N. J.

Couple stresses are associated with couples per unit area. For materials in which they are present, the usual stress tensor is not symmetric. Couple-stress effects in linear elasticity have been studied by Mindlin and Tiersten,[1] Mindlin,[2] and Sadowsky, Hsu, and Hussain.[3] These authors noted that, generally, the main effects of couple stresses are to be found in thin layers adjacent to the boundary curves of the problem. For example, Mindlin's exact solution for a stress-free circular hole in a plate subjected to a uniform stress at infinity clearly exhibits this boundary-layer-type behavior.

The pertinent parameter for couple-stress effects is $\epsilon = l/a$ where l is the couple-stress material constant and a is a characteristic length of the problem. The linear couple-stress elastic theory is valid for $\epsilon < 1$.[1] It appeared to the first author that couple-stress problems with small ϵ can be treated quite generally as a singular perturbation problem by following the formalism of the method of inner-outer expansion. For general references to this method, see Van Dyke.[4] In essence, we take advantage of the fact that ϵ is small and proceed to construct an approximate solution that remains uniformly valid in the limit $\epsilon \rightarrow 0$. This method can readily be applied to couple-stress problems with arbitrary geometries and boundary conditions. The generality of the method should compensate more than adequately for the loss of accuracy. For example, the elliptic-hole version of Mindlin's problem could be very simply treated, whereas an attempt on an exact solution proved to be a formidable task.[5] The present authors have developed the general inner-outer expansion solution including the leading terms for couple-stresses for a hole of arbitrary shape. These results will appear in a later publication. In the present note, we shall illustrate the method of inner-outer expansion by applying it to the simple case of a circular hole in a plate under uniform tension at infinity.

From Mindlin,[2] the potential functions φ and ψ satisfies the following equations:

$$\nabla^4 \varphi = 0 \tag{1}$$

$$\nabla^2\psi - \epsilon^2\nabla^4\psi = 0 \tag{2}$$

$$\frac{\partial}{\partial r}(\psi - \epsilon^2 \nabla^2\psi) = -2(1-\nu)\,\epsilon^2 \frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2\varphi \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(\psi - \epsilon^2 \nabla^2\psi) = 2(1-\nu)\,\epsilon^2 \frac{\partial}{\partial r}\nabla^2\varphi \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{5}$$

and all variables have been properly nondimensionalized. Note that Eqs. (1–4) are not independent, for Eqs. (1) and (2) can be derived from Eqs. (3) and (4). The usual stress components and the couple-stress components are derived from the potentials by means of

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} \\ \sigma_\theta &= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta} \\ \tau_{r\theta} &= -\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \\ \tau_{\theta r} &= -\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\partial^2\psi}{\partial r^2} \\ \mu_r &= \frac{\partial\psi}{\partial r} \\ \mu_\theta &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} \end{aligned}\right\} \tag{6}$$

where $\tau_{\theta r}$ is the shear stress in the r direction on the θ-constant surface and similarly for $\tau_{r\theta}$. μ_r is the couple-stress in z direction on the r constant surface and similarly for μ_θ. The boundary conditions at infinity are

$$\sigma_x = 1 \quad \text{at} \quad r \rightarrow \infty \tag{7}$$

and all other stresses and couple-stresses vanish. The boundary conditions on the boundary of the hole are

$$\sigma_r = \tau_{r\theta} = \mu_r = 0 \quad \text{at} \quad r = 1 \tag{8}$$

We begin by saying that the solution for φ and ψ can be expanded in the following form:

$$\begin{aligned} \varphi &= \Phi_0(\xi,\theta) + \epsilon\Phi_1(\xi,\theta) + \epsilon^2\Phi_2(\xi,\theta) + \dots \\ \psi &= \Psi_0(\xi,\theta) + \epsilon\Psi_1(\xi,\theta) + \epsilon^2\Psi_2(\xi,\theta) + \dots \end{aligned} \tag{9}$$

where $\xi = r - 1$. The preceding expansions are called the

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* Associate Professor, Department of Aerospace and Mechanical Sciences. Member AIAA.

† Assistant Professor, Department of Aerospace and Mechanical Sciences. Member AIAA.

outer expansions and are assumed valid away from the rim of the hole. Around the rim of the hole, we assume that the solution for φ and ψ are to be expanded in a different manner:

$$\begin{aligned} \Phi &= \varphi_0(\xi^*, \theta) + \epsilon\varphi_1(\xi^*, \theta) + \epsilon^2\varphi_2(\xi^*, \theta) + \ldots \\ \psi &= \psi_0(\xi^*, \theta) + \epsilon\psi_1(\xi^*, \theta) + \epsilon^2\psi_2(\xi^*, \theta) + \ldots \end{aligned} \tag{10}$$

which are called the inner expansions. The new variable ξ^* is defined by

$$\xi^* = \xi/\epsilon \tag{11}$$

The choice of this new variable is motivated by the realization that in the inner region the couple-stress terms in the differential equations are to be of the same order of magnitude (with respect to ϵ) as the other classical terms in the limit as $\epsilon \to 0$.

By substituting Eqs. (9) and (10) into Eqs. (1) to (4), and collecting terms of like powers in ϵ, we obtain a system of partial differential equations for Φ_n and Ψ_n, the various order terms of the outer expansions, and a system of ordinary differential equations for φ_n and ψ_n, the various order terms of the inner expansions. The boundary conditions at infinity are to be satisfied by the outer expansions, whereas the boundary conditions at the hole, Eq. (8), are to be satisfied by the inner expansions. Furthermore, we require, following the formalism of the method of inner-outer expansion, that the asymptotic expansions for the inner solution as $\xi^* \to \infty$ are to match analytically the Taylor series expansions of the outer solution near $\xi \to 0$. It is through this matching procedure that the inner and outer solutions interact with each other. For example, one of the matching conditions is

$$\lim_{\xi^*\to\infty} \frac{\partial}{\partial r}\psi_n(\xi^*, \theta) = \frac{(\xi^*)^n}{n!}\Psi_{0,n+1}(\theta) + \frac{(\xi^*)^{n-1}}{(n-1)!}\Psi_{1,n}(\theta) + \ldots \Psi_{n,0} \tag{12}$$

where $\Psi_{m,n}(\theta)$ is defined by

$$\Psi_m(\xi, \theta) = \Psi_{m0}(\theta) + \ldots + \frac{\epsilon^n}{n!}\Psi_{mn}(\theta) + \ldots \tag{13}$$

By means of simple quadratures, the following solutions for φ and ψ are obtained:

$$\left.\begin{aligned}
&\psi_0 = 0 \qquad \Psi_0 = 0 \qquad \psi_1 = 0 \qquad \Psi_1 = 0 \\
&\psi_2 = 2(1-\nu)\int(\Phi_{02} + \Phi_{03})d\theta \\
&\Psi_2 = 4(1-\nu)(1/r^2)\sin 2\theta \\
&\psi_3 = -2(1-\nu)\Phi_0{}'_2(e^{-\xi^*} + \xi^*) \\
&\Psi_3 = 0 \\
&\psi_4 = -(1-\nu)\Phi_0{}'_3\xi^{*2} + (1-\nu)\Phi_0{}'_2 e^{-\xi^*} \times (\xi^* + 1) + \Psi_4{}'_0 \\
&\varphi_4 = \tfrac{1}{24}\Phi_{04}\xi^{*4} + \tfrac{1}{2}\Phi_{22}\xi^{*2} + [2(1-\nu)\Phi_0{}'_2{}' - \Phi_3{}'_0{}']\xi^* + \Phi_{40} \\
&\varphi_0 = \Phi_{00} \\
&\Phi_0 = \tfrac{1}{2}\cos 2\theta + \tfrac{1}{4}r^2(1-\cos 2\theta) - \tfrac{1}{2}\ln r - (1/4r^2)\cos 2\theta \\
&\varphi_1 = -\Phi_0{}'_0{}'\xi^* \qquad \Phi_1 = 0 \qquad \varphi_2 = \tfrac{1}{2}\Phi_{02}\xi^{*2} + \Phi_{20} \\
&\Phi_2 = -4(1-\nu)[1 - (1/r^2)]\cos 2\theta \\
&\varphi_3 = \tfrac{1}{6}\Phi_{03}\xi^{*3} + \Phi_{21}\xi^* + \Phi_{30} \\
&\Phi_3 \text{ not yet required}
\end{aligned}\right\} \tag{14}$$

Note that Φ_0 is the classical noncouple stress solution.

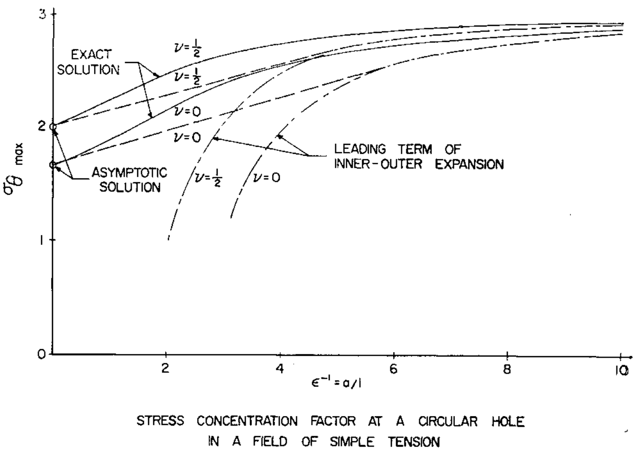

Fig. 1 Stress concentration factor at a circular hole in a field of simple tension.

Finally we look at the distribution of the stress component σ_θ by examining the first three terms of the inner expansion:

$$\begin{aligned} \sigma_{\theta_0} &= 1 - 2\cos 2\theta \\ \sigma_{\theta_1} &= (-1 + 6\cos 2\theta)\xi^* \\ \sigma_{\theta_2} &= (\tfrac{3}{2} - 15\cos 2\theta)\xi^{*2} + (16(1-\nu)\cos 2\theta)e^{-\xi^*} \end{aligned} \tag{15}$$

where

$$\sigma_\theta = \sigma_{\theta_0}(\xi^*, \theta) + \epsilon\sigma_{\theta_1}(\xi^*, \theta) + \epsilon^2\sigma_{\theta_2}(\xi^*, \theta) + \ldots \tag{16}$$

The leading term in the couple-stress effect is the second term of σ_{θ_2}, which exhibits a singular behavior in the form $\exp(-\xi/\epsilon)$. The maximum stress concentration in the classical noncouple-stress case is given by σ_{θ_0} at $\xi^* = 0$ and $\theta = \pi/2$, namely, $\sigma_{\theta\max} = 3$. The couple stress theory reduces the maximum stress by $16(1-\nu)\epsilon^2$, i.e.,

$$\sigma_{\theta\max} = 3 - 16(1-\nu)\epsilon^2 + \ldots \tag{17}$$

Although the linear couple-stress elastic theory is not valid for ϵ near or greater than unity, the mathematical solution for large ϵ can be obtained as an ordinary expansion in ϵ^{-1}. We have found the asymptotic solution which yields for $\sigma_{\theta\max}$ the value

$$\lim_{\epsilon\to\infty} \sigma_{\theta\max} = (5 - 2\nu)/(3 - 2\nu) \tag{18}$$

Although physically unrealistic, the asymptotic solution along with the leading term of the inner-outer expansion allows one to construct a function for $\sigma_{\theta\max}$ which is a good approximation not only for very small values of ϵ but also for values that are not so small.

We have constructed such curves in Fig. 1 by use of the leading term of the inner-outer expansion and a straight line that is tangent to the inner-outer expansion curve and has the value assigned by Eq. (18) at $\epsilon^{-1} = 0$. The approximate curves can be compared with Mindlin's exact solution in Fig. 1.

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